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Non-linear approximations for natural convection in a horizontal annulus

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Abstract

Two different problems are proposed as approximations of the usual system modelling natural convection under the Oberbeck–Boussinesq assumptions. The error is evaluated by means of the norm of its gradient in the Hilbert space. The average Nusselt number is also estimated.

Keywords: Oberbeck–Boussinesq approximation; Natural convection; Nusselt number

1. Introduction

Natural convection heat transfer in horizontal coaxial cylinders with uniform temperature boundary conditions on the inner and outer surfaces, respectively, has gained considerable attention owing to its wide applications (solar concentrators, energy storage systems, cooling systems in nuclear reactors, electrical gas insulated transmission lines etc.).

A comprehensive review covering available experimental data, numerical results, mathematical models and correlations for heat transfer in a horizontal circular annulus has been presented by Teerstra and Yovanovich [1], whose analysis is in terms of the average Nusselt number, expressed as a function of the Rayleigh number Ra and of the aspect ratio R_2/R_1.

The average heat transfer rate in the annulus shows two limiting cases connected by an intermediate transition region. For small values of Ra or of R_2/R_1, the average Nusselt number shows a strong dependence on R_2/R_1, while it is approximately independent of Ra; this behaviour is termed conductive. At large Ra and large R_2/R_1 boundary layer behaviour occurs. In this region the average Nusselt number shows a strong dependence on Ra but a somewhat weaker dependence on R_2/R_1.

The flow patterns observed in the horizontal circular annulus have been classified by Powe et al. [2]. They found four basic types of flow, depending upon Ra and the inverse relative gap width $\mathscr{A} = 2R_1/(R_2 - R_1)$. For sufficiently small Ra and for any $\mathscr{A}$, the flow observed is a steady 2-D one, with two crescent-shaped eddies which are symmetric with respect to the vertical plane through the common axis of the cylinders. As Ra increases above a critical value, different unsteady flow patterns were observed: 2-D oscillatory flow for $\mathscr{A} < 2.8$ (wide gap), 3-D spiral flow for $2.8 < \mathscr{A} < 8.5$, 2-D multi-cellular flow for $\mathscr{A} > 8.5$ (narrow gap).

More recently, the flow pattern analysis has been examined closely by other authors, typically by Yoo [3] with his connected references. Yoo reports that if Ra is greater than a certain critical value, dual steady solutions are realized in the flow regime in which Powe [2] asserts that a steady 2-D flow prevails. One solution corresponds to the commonly observed crescent-shaped pattern in which the fluid ascends along the central plane when the inner cylinder is kept hotter, and the other is a flow consisting of two counter-rotating eddies and their mirror images.

This corresponds to a first transition with exchange of stability, from the basic shear driven upward flow, which for $Ra > 0$ is not the rest state but is steady, to a buoyancy driven downward flow, steady as well.

In this flow pattern analysis, all Authors agree on the assertion that for sufficiently small Ra, and independently of $\mathscr{A}$, the steady unicellular crescent-shaped eddies occur.

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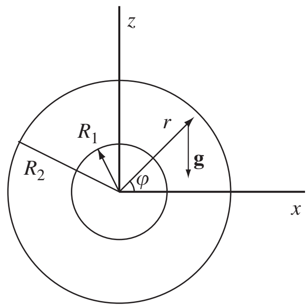

Fig. 1. ϕ is the counterclockwise angle from the horizontal x-axis, r the radial distance from the origin and z the vertical axis. The boundary conditions on the temperature are: $T(R_1, \phi) = T_1$ and $T(R_2, \phi) = T_2 < T_1$.

It is exactly in this region that we analyze some mathematical models for natural convection in horizontal annuli from the theoretical point of view. We pay particular attention to the contribution to the heat transfer given by the motion, since as we have just mentioned the fluid moves for any difference of temperature $\Delta T = (T_1 - T_2) > 0$, no matter how small. This means that all the mathematical models describe the situation as follows: if one substitutes velocity $\mathbf{v} = 0$ in the equations of motion, then it is not possible to solve the system of equations with a corresponding temperature field T.

By considering the region in which stable basic steady flows occur, in [5] it can be seen how the linear Stokes-like system, previously studied in [6], works as approximation of the basic conservation laws under the Oberbeck Boussinesq simplification. However, still in [5] it is proved that by using such a model, although there is fluid motion, the heat transfer is the same as for conduction.

The present paper deals with the validity of the non-linear Stokes problem [7] and of another almost linear system (which we call *uncoupled problem*) as approximations of the full system of equations.

To derive them, a scheme of the physical system under examination is shown in Fig. 1. Our 2D-domain is endowed with a reference frame with x- and z-axis, and polar coordinates are chosen as in the picture. Accordingly, $\mathbf{e}_3 = \mathbf{k} = \nabla(r \sin \varphi) = \sin \varphi \mathbf{e}_r + \cos \varphi \mathbf{e}_\varphi$.

As usual, the basic conservation laws for a Newtonian fluid are used. Next, the equations are simplified by means of the Oberbeck–Boussinesq approximation, whose proper assumptions are still the subject of discussion [8,9]. Further, the unknown $\tau = T - T^*$, where

$$T^* = T_1 + \frac{\Delta T}{\ln R_2 - \ln R_1}(\ln R_1 - \ln r)$$

with $\Delta T = T_1 - T_2$, is introduced to get homogeneous boundary conditions in the energy equation. Notice that T^* is the temperature field driven by pure conduction.

The steady version of the full system is then

$$\nabla \cdot \mathbf{v} = 0, \tag{1.1}$$

$$\frac{1}{Pr}\mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \Pi + \Delta \mathbf{v} + \frac{Ra}{b} \sin \varphi \, \mathbf{e}_r + Ra\tau\nabla(r \sin \varphi), \tag{1.2}$$

$$\mathbf{v} \cdot \nabla \tau - \frac{v_r}{rb} = \Delta \tau, \tag{1.3}$$

where $b := \ln(R_2/R_1) = \ln(1 + 2/\mathscr{A})$. Boundary conditions are appended to the system

$$v_r(\mathscr{A}/2, \varphi) = 0, \quad v_\varphi(\mathscr{A}/2, \varphi) = 0,$$

$$v_r(\mathscr{A}/2 + 1, \varphi) = 0, \quad v_\varphi(\mathscr{A}/2 + 1, \varphi) = 0,$$

$$\tau(\mathscr{A}/2, \varphi) = 0, \quad \tau(\mathscr{A}/2 + 1, \varphi) = 0.$$

Moreover, all the functions must be periodic with respect to φ, otherwise the solution does not physically make sense.

Length, time and temperature used herein are dimensionless, being related to the real variables by the equations

$$t = \frac{\alpha}{\delta^2}t', \quad r = \frac{r'}{\delta}, \quad z = \frac{z'}{\delta}, \quad \tau = \frac{\tau'}{\Delta T},$$

where the reference length δ is $R_2 - R_1$ and α is the thermal diffusivity.

With such units, the domain is

$$\Omega_{\mathscr{A}} := \{(r, \varphi) \in \mathbb{R}^2 : r \in (\mathscr{A}/2, 1 + \mathscr{A}/2), \varphi \in (-\pi, +\pi)\}.$$

By denoting g as the gravitational acceleration, ν the kinematic viscosity and β the volumetric expansion coefficient, the Prandtl and Rayleigh numbers in Eq. (1.2) are

$$Pr = \frac{\nu}{\alpha}, \quad Ra = \frac{\beta g \Delta T \delta^3}{\nu\alpha}.$$

We make here some remarks about the notation. In the present paper boldfaced small letters always stand for vectors and vector-valued functions. As happens in fluid dynamics, by $\mathbf{u} \cdot \nabla \mathbf{w}$ we mean $u^j \partial_j \mathbf{w}$. We indicate by using the same symbol $\| \cdot \|_q$ the L^q-norm of scalar-, vector- and tensor-valued functions. If the subscript q is not present, we mean the norm in the Hilbert space.

Several simplified versions of problem (1.1)–(1.3) can be considered and discussed in terms of the three dimensionless parameters: Ra, $\mathscr{A}$ and Pr. The system can in principle be simplified by neglecting the first or both the non-linear terms $\mathbf{v} \cdot \nabla \mathbf{v}$ and $\mathbf{v} \cdot \nabla \tau$. If both are neglected, the result is the Stokes-like problem, addressed in [5].

On the other hand, a very used approximation of (1.1)–(1.3), see for example [7], is the *non-linear Stokes problem* obtained by neglecting $\mathbf{v} \cdot \nabla \mathbf{v}$, but retaining $\mathbf{v} \cdot \nabla \tau$. Although the truly linear Stokes-like problem could sometimes be used for computations [6], in the present paper it is shown that the almost linear one works in a larger region of parameter space.

The non-linear Stokes problem is the following:

$$\nabla \cdot \mathbf{v}_0 = 0, \tag{1.4}$$

$$0 = -\nabla \Pi_0 + \Delta \mathbf{v}_0 + \frac{Ra}{b} \sin \varphi \mathbf{e}_r + Ra\tau_0\nabla(r \sin \varphi), \tag{1.5}$$

$$\mathbf{v}_0 \cdot \nabla \tau_0 - \frac{v_0^r}{rb} = \Delta \tau_0. \tag{1.6}$$

In Section 2, we prove that it can be thought as a good approximation of (1.1)–(1.3): (a) for fixed Pr and $\mathscr{A}$, as Ra goes to 0, (b) for fixed Ra and $\mathscr{A}$, as Pr goes to infinity.

In more detail: the velocity and temperature fields which are solutions of (1.1)–(1.3) and (1.4)–(1.6), and the difference between such solutions, are of the order of $f_1(\mathscr{A}) Ra\, b^{-1}$ for the solutions, and $f_2(\mathscr{A}) Ra^2 b^{-2} Pr^{-1}$ for their differences (f_1 and f_2 are continuous functions). The Dirichlet norm of the difference identifies the absolute error. Thus, it is expected that the relative error tends to 0 in both cases (a) and (b).

Furthermore, in the present paper the estimates given in [5] are refined for $\mathscr{A}$ close to 0, by obtaining a wider interval for the parameter Ra.

Next, another simplified problem is introduced: the one in which not only the convective term but also the coupling term $Ra\tau\nabla(r \sin\varphi)$ are neglected in the momentum equation (1.1)

$$\nabla \cdot \mathbf{v}_0 = 0, \tag{1.7}$$

$$0 = -\nabla \Pi_0 + \Delta \mathbf{v}_0 + \frac{Ra}{b} \sin\varphi\, \mathbf{e}_r, \tag{1.8}$$

$$\mathbf{v}_0 \cdot \nabla \tau_0 - \frac{v_0^r}{rb} = \Delta \tau_0. \tag{1.9}$$

We show that such an *uncoupled problem* is also a good approximation of the full problem for small Ra/b, with fixed Pr and $\mathscr{A}$. But it does not work for large Pr with fixed Ra and $\mathscr{A}$, because the difference between (1.1)–(1.3) and (1.7)–(1.9) does not depend on Pr^{-1} as a factor.

The solution of (1.7)–(1.9) necessarily exhibits non-trivial symmetry features, which are approximately verified also by numerical solutions known in the literature [6]. We are referring to the case of symmetry under reflection with respect to the horizontal axis, which is approximately verified by numerical solutions of the complete problem for small Ra.

A further remarkable mathematical property of (1.7)–(1.9) is that unlike (1.1)–(1.3) and (1.4)–(1.6), the existence of solutions can immediately be proved without any restriction on the size of Ra.

Finally, in Section 3 we give estimates of the average Nusselt number

$$\overline{Nu}_\delta(T) = \frac{\delta}{\Delta T\, A} \int_A \nabla T \cdot \mathbf{n}\, \mathrm{d}A, \tag{1.10}$$

where A can be the inner or the outer surface. In the systems (1.4)–(1.6) and (1.7)–(1.9), we have $T = T^* + \tau_0$ and consequently, by choosing A as the outer surface

$$\overline{Nu}_\delta(T) = \frac{1}{b(1 + \mathscr{A}/2)} + \overline{Nu}_\delta(\tau_0).$$

The quantity $\overline{Nu}_\delta(\tau_0)$ measures the heat transfer due to the flows described by means of solutions of (1.4)–(1.6) or (1.7)–(1.9).

In order to be fully appropriate, an approximation should only cause small errors in the average Nusselt number. Then, if compared with the linear Stokes problem, the problems studied herein have the following different features: on one hand, they can be used as approximations in a larger region of the space of dimensionless parameters, and on the other hand the quantity $\overline{Nu}_\delta(\tau_0)$ is different from 0.

Therefore by considering both the simplified systems, the estimates of $\overline{Nu}_\delta(\tau) - \overline{Nu}_\delta(\tau_0)$ (where τ solves the full problem) are derived, and the absolute error is even smaller than for velocity and temperature.

2. Non-linear approximations: a theoretical justification

If steady solutions of problem (1.1)–(1.3) are looked for, it can be seen in [5] that a numerical bound for their norms (the so-called *a priori* estimates, needed to get existence theorems) is found only through restrictions on the Rayleigh number. Since they are only sufficient conditions, the domain of existence for steady solutions could be larger. In any case, we can provide for such values of Ra the approximations (1.4)–(1.6) and (1.7)–(1.9).

Furthermore, unlike as in [5], the approach of the present paper deserves specific attention to be paid also to the range of small $\mathscr{A}$, so that the restrictions on Ra can considerably be weakened in this range.

To this end, two general estimates for the Poincaré constant are compared: the one given in [5]

$$k_p^1 = \frac{1}{2}\sqrt{1 + \frac{2}{\mathscr{A}}}, \tag{2.1}$$

and the traditional one

$$k_p^s = 1 + \mathscr{A}/2, \tag{2.2}$$

which can be found for instance in [10]. As can immediately be seen, the second expression is more suitable for $\mathscr{A} < (\sqrt{2} - 1)$, while the first one is better for large $\mathscr{A}$.

Of course, the optimal value of k_p (which is $(-\lambda_0)^{-1/2}$, with λ_0 the largest eigenvalue of the Laplacian for scalar functions, and of the Stokes operator for divergence free vector fields) could be found for any $\Omega_{\mathscr{A}}$, but only through a numerical analysis which would exclude general evaluations.

In what follows, the bounds of some norms of the solutions are considered in Hilbert space. Thus, we just mention that since the boundary is regular, it can be proved in a general way that any solution of our problems is regular. In particular, in order to find a solution of the three systems of equations given in the introduction, we can apply the following standard scheme: first an *a priori* bound is found for the norm of the gradient of the unknown velocity and temperature fields, then the existence of a so-called *weak-solution* can be proved, and finally it follows that the solution is also bounded pointwise and the norms in the space of the continuous functions should fulfil analogous estimates. Moreover, the pressure field can *a posteriori* be found. In the present paper any proof of existence is omitted for the sake of brevity, since the main work consists in writing and discussing the error estimates. In any case, existence proofs can be found in [11].

By considering the solutions of (1.1)–(1.3) and (1.4)–(1.6), the following proposition can be written:

Proposition 1. *Let us assume*

$$\delta_0 := 1 - \frac{2(k_p^1)^4 Ra}{\mathscr{A} b} > 0, \tag{2.3}$$

then any solution $(\mathbf{v}, \tau)$ of problem (1.1)–(1.3), *and any solution* $(\mathbf{v}_0, \tau_0)$ *of the non-linear Stokes problem* (1.4)–(1.6) *corresponding to the same* $\mathscr{A}$ *and Ra*, *verify the same estimates*, *which we write as follows*

$$\|\nabla \mathbf{v}\|, \|\nabla \mathbf{v}_0\| \leqslant \sqrt{\pi\left(\frac{1+\mathscr{A}}{2}\right)} \frac{k_p^1 Ra}{b\delta_0}, \tag{2.4}$$

$$\|\nabla \tau\|, \|\nabla \tau_0\| \leqslant \sqrt{\pi\left(\frac{1+\mathscr{A}}{2}\right)} \frac{2(k_p^1)^3 Ra}{\mathscr{A} b^2 \delta_0}. \tag{2.5}$$

Moreover, *if*

$$\delta_1 := 1 - \sqrt{\frac{\pi}{b}} (k_p^{\mathrm{s}})^3 Ra > 0, \tag{2.6}$$

then the following estimates also hold

$$\|\nabla \mathbf{v}\|, \|\nabla \mathbf{v}_0\| \leqslant \sqrt{\pi\left(\frac{1+\mathscr{A}}{2}\right)} \frac{k_p^{\mathrm{s}} Ra}{b\delta_1}, \tag{2.7}$$

$$\|\nabla \tau\|, \|\nabla \tau_0\| \leqslant \pi \sqrt{\frac{1+\mathscr{A}}{2}} \frac{(k_p^{\mathrm{s}})^2 Ra}{\sqrt{b^3}\delta_1}. \tag{2.8}$$

Proof. The first part of the statement, with inequalities (2.4) and (2.5), is identical to Proposition 1 in [5] and is proved there.

In the same way we prove inequalities (2.7) and (2.8). The difference between the two parts of the statement all lies in the manipulation of the energy equation. Precisely, in order to get (2.7) and (2.8), after having multiplied (1.3) as well as (1.6) by τ, we must integrate over $\Omega_{\mathscr{A}}$ and use the estimate

$$\frac{1}{b}\left| \int \frac{v_r}{r}\tau \right| \leqslant \sqrt{\frac{\pi}{b}} k_p^{\mathrm{s}} \|\nabla \tau\| \|\nabla \mathbf{v}\|, \tag{2.9}$$

whose proof is shown in Appendix A. Hence, from (2.9) it can be deduced that

$$\|\nabla \tau\| \leqslant \sqrt{\frac{\pi}{b}} k_p^{\mathrm{s}} \|\nabla \mathbf{v}\|. \tag{2.10}$$

For small $\mathscr{A}$, (2.10) is finer than

$$\|\nabla \tau\| \leqslant \frac{2(k_p^1)^2}{\mathscr{A} b} \|\nabla \mathbf{v}\|,$$

which is the one from which the first part of the statement follows. Conversely, this last is finer for $\mathscr{A} \to \infty$, since $\mathscr{A} b$ tends to 2 while b itself tends to 0. At the same time, condition (2.6), which implies the second part of the statement, is less restrictive for $\mathscr{A}$ close to 0.

Now, we can analyze the error in the approximation by writing estimates for the difference between solutions of the two systems.

Thus, after having set

$$\gamma = \sqrt{\pi\left(\frac{1+\mathscr{A}}{2}\right)},$$

a proposition is given here summarizing the outcomes. Again, the proof can be found in Appendix A.

Proposition 2. *Let us fix Ra*, *Pr and* $\mathscr{A}$. *Given a solution* $(\mathbf{v}, \tau)$ *of problem* (1.1)–(1.3) *and a solution* $(\mathbf{v}_0, \tau_0)$ *of* (1.4)–(1.6), *we set*

$$\mathbf{v} = (\mathbf{v} - \mathbf{v}_0) + \mathbf{v}_0 := \mathbf{u} + \mathbf{v}_0,$$

$$\tau = (\tau - \tau_0) + \tau_0 = \sigma + \tau_0. \tag{2.11}$$

Then, *if*

$$\alpha_0 := \delta_0 - \frac{(k_p^1)^2}{b\sqrt{2}} \frac{\gamma}{\delta_0} Ra \left(\frac{1}{Pr} + 1 - \delta_0\right) > 0 \tag{2.12}$$

the following estimates for the difference are verified:

$$\|\nabla \mathbf{u}\| \leqslant \frac{1}{\alpha_0} \frac{(k_p^1)^3 \gamma^2}{\sqrt{2}\delta_0^2} \frac{1}{Pr} \frac{Ra^2}{b^2},$$

$$\|\nabla \sigma\| \leqslant \frac{1}{\alpha_0} \frac{\sqrt{2}(k_p^1)^5 \gamma^2}{\mathscr{A} b \delta_0^2} \left(1 + \frac{(k_p^1)^2 Ra\gamma}{b\sqrt{2}\delta_0}\right) \frac{1}{Pr} \frac{Ra^2}{b^2}, \tag{2.13}$$

and if

$$\alpha_1 := \delta_1 - \frac{(k_p^{\mathrm{s}})^2}{b\sqrt{2}} \frac{\gamma}{\delta_1} Ra \left(\frac{1}{Pr} + 1 - \delta_1\right) > 0 \tag{2.14}$$

the following estimates are verified:

$$\|\nabla \mathbf{u}\| \leqslant \frac{1}{\alpha_1} \frac{(k_p^{\mathrm{s}})^3 \gamma^2}{\sqrt{2}\delta_1^2} \frac{1}{Pr} \frac{Ra^2}{b^2},$$

$$\|\nabla \sigma\| \leqslant \frac{1}{\alpha_1} \frac{(k_p^{\mathrm{s}})^4 \gamma^2}{\sqrt{2}\delta_1^2} \sqrt{\frac{\pi}{b}} \left(1 + \frac{(k_p^{\mathrm{s}})^2 Ra\gamma}{\sqrt{2} b \delta_1}\right) \frac{1}{Pr} \frac{Ra^2}{b^2}. \tag{2.15}$$

Conditions (2.12) and (2.14) are possible further smallness conditions on *Ra*, with respect to (2.3) and (2.6). Therefore, some remarks on their meaning should be done, while a detailed discussion in terms of the parameters is just a matter of elementary analysis.

Actually, (2.12) and (2.14) are needed to write the estimates of the error, and hence are necessary to define the domain where problem (1.4)–(1.6) is a good approximation of problem (1.1)–(1.3). As can be immediately verified, the largest admissible *Ra* for fixed *Pr* and $\mathscr{A}$ is smaller than that implied by (2.3) and (2.6).

In conclusion, by comparing the estimates of Proposition 2 with those of Proposition 1, it can be seen that for sufficiently small Ra the error is of order Ra^2 while the solution is of order Ra. Also, the error goes to 0 as Pr goes to infinity while the estimate of the solution does not depend on Pr.

Further, let us fix Pr and consider the estimates for $\mathscr{A}$ close to 0. Since for small $\mathscr{A}$ it immediately follows that b is large, conditions (2.6) and (2.14) provide as largest admissible Ra a function of $\mathscr{A}$ which is non-uniformly bounded. Moreover, estimates (2.7) and (2.8) can be read as depending on Ra/b, while (2.15) can be read as depending on Ra^2/b^2. Therefore, for small $\mathscr{A}$ and fixed Pr the order of the relative error can be small even when Ra is not small. Thus, the region of parameter space in which the approximation can be considered is larger than in [5].

Finally, the decoupled problem (1.7)–(1.9) is analyzed. A specific consequence of erasing the coupling term is that for all $Ra > 0$, after having performed the scalar product of the momentum equation with $\mathbf{v}_0$, and after having applied Schwartz and Poincaré inequalities, the norm of the gradient of the velocity can be estimated, so that no restrictions on the parameters arise.

Another consequence is the symmetry of the velocity field with respect to the horizontal axis. This is evident in the stream-function formulation of the momentum equation, which for the reference system given in Fig. 1 is

$$\Delta^2\Psi = Ra\,\frac{1}{br}\cos\varphi,$$

where

$$v_0^r = -\frac{1}{r}\frac{\partial\Psi}{\partial\varphi};\quad v_0^\varphi = \frac{\partial\Psi}{\partial r}.$$

So, it follows that v_0^r and v_0^φ are, respectively, odd and even functions with respect to φ.

The system (1.7)–(1.9) and the error in the approximation can be addressed by means of the same technique as used in the proof of the previous statements. Therefore, the proof of Proposition 3 is omitted.

Proposition 3. *A solution of* (1.7)–(1.9) *always exists and is bounded as follows*:

$$\|\nabla\mathbf{v}_0\| \leqslant \frac{k_p}{b}\,\gamma Ra;\quad \|\nabla\tau_0\| \leqslant \frac{2k_p^3}{\mathscr{A}b^2}\,\gamma Ra. \tag{2.16}$$

For the temperature excess, the estimate

$$\|\nabla\tau_0\| \leqslant k_p^2\sqrt{\frac{\pi}{b^3}}\gamma Ra, \tag{2.17}$$

can be used and k_p can be both k_p^{l} or k_p^{s}. Then, we can say that if

$$\eta_0 := \delta_0 - \frac{(k_p^{\mathrm{l}})^2}{b\sqrt{2}}\gamma Ra\left(\frac{1}{Pr} + 1 - \delta_0\right) > 0 \tag{2.18}$$

the difference from the solution of the complete problem, defined as in (1.7)–(1.9), *is bounded as follows*:

$$\|\nabla\mathbf{u}\| \leqslant \frac{(k_p^{\mathrm{l}})^3}{\eta_0}\gamma\left(\frac{\gamma}{\sqrt{2}Pr} + \frac{2(k_p^{\mathrm{l}})^2}{\mathscr{A}}\right)\frac{Ra^2}{b^2}, \tag{2.19}$$

$$\|\nabla\sigma\| \leqslant \frac{1}{\eta_0}\frac{2(k_p^{\mathrm{l}})^5}{\mathscr{A}b}\gamma\left(\frac{(k_p^{\mathrm{l}})^2 Ra\gamma}{2\sqrt{2}} + 1\right) \times\left(\frac{\gamma}{\sqrt{2}Pr} + \frac{2(k_p^{\mathrm{l}})^2}{\mathscr{A}}\right)\frac{Ra^2}{b^2}. \tag{2.20}$$

Moreover, if

$$\eta_1 := \delta_1 - \frac{(k_p^{\mathrm{s}})^2}{b\sqrt{2}}\gamma Ra\left(\frac{1}{Pr} + 1 - \delta_1\right) > 0 \tag{2.21}$$

the following also holds true

$$\|\nabla\mathbf{u}\| \leqslant \frac{(k_p^{\mathrm{s}})^3}{\eta_1}\gamma\left(\frac{\gamma}{\sqrt{2b}Pr} + k_p^{\mathrm{s}}\sqrt{\pi}\right)\frac{Ra^2}{b^3}, \tag{2.22}$$

$$\|\nabla\sigma\| \leqslant \frac{(k_p^{\mathrm{s}})^4}{\eta_1}\gamma\sqrt{\pi}\left(\frac{(k_p^{\mathrm{s}})^2 Ra\gamma}{b\sqrt{2}} + 1\right) \times\left(\frac{\gamma}{b\sqrt{2Pr}} + k_p^{\mathrm{s}}\sqrt{\frac{\pi}{b}}\right)\frac{Ra^2}{b^3}. \tag{2.23}$$

3. Estimates for the Nusselt number

Since we are working in 2-D, all the problems studied can be rewritten by means of the stream function Ψ. Thus, the relationship with the solenoidal velocity fields is given by

$$v_0^r = -\frac{1}{r}\frac{\partial\Psi}{\partial\varphi};\quad v_0^\varphi = \frac{\partial\Psi}{\partial r}.$$

Once dimensionless variables are taken into account, to calculate the Nusselt number averaged over the outer surface (see (1.10) for the definition) we actually need to consider the quantity

$$\overline{Nu}_\delta(\tau_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial\tau_0}{\partial r}(1 + \mathscr{A}/2, \varphi)\,\mathrm{d}\varphi. \tag{3.1}$$

If we want to evaluate such a quantity, the periodicity property of the solution should be used: in fact, problems (1.4)–(1.6) and (1.7)–(1.9) can be faced by expressing their solutions in suitable bases, and the angular dependence can always be written by using a Fourier series. For instance, for Ψ the eigenfunctions of the bilaplacian with gradient vanishing on the boundary can be chosen, and for τ_0 the eigenfunctions of the Laplacian which also vanish on the boundary. Their dependence on φ will be always a combination of $\sin n\varphi$ and $\cos n\varphi$ with integer n.

Therefore, we can write

$$\Psi = f_1(r) + \Psi^{(\mathrm{e})}(r,\varphi) + \Psi^{(\mathrm{o})}(r,\varphi),$$

$$\tau_0 = f_2(r) + \tau_0^{(\mathrm{e})}(r,\varphi) + \tau_0^{(\mathrm{o})}(r,\varphi),$$

where (e) and (o) stand for even and odd, with respect to the x-axis in Fig. 1.

We have

$$\int_{-\pi}^{\pi} \frac{\partial \tau_0}{\partial r}(1+\mathscr{A}/2, \varphi)\,\mathrm{d}\varphi = 2\pi f_2'(1+\mathscr{A}/2).$$

We also know that since $f_2(\mathscr{A}/2) = f_2(1+\mathscr{A}/2) = 0$, then a value $r^* \in (\mathscr{A}/2, 1+\mathscr{A}/2)$ of the radius exists such that $f_2'(r^*) = 0$.

Further information can be obtained from the energy equation which is

$$\Delta\tau_0 = \frac{1}{br^2}\frac{\partial \Psi}{\partial \varphi} + \mathbf{v}_0 \cdot \nabla\tau_0. \tag{3.2}$$

We integrate over $\varphi \in (-\pi, +\pi)$ all terms in (3.2). So, in particular we obtain

$$\int_{-\pi}^{\pi} \Delta\tau_0\,\mathrm{d}\varphi = \int_{-\pi}^{\pi} \Delta f_2\,\mathrm{d}\varphi = 2\pi(f_2'' + f_2'/r),$$

$$\int_{-\pi}^{\pi} \frac{1}{br^2}\frac{\partial \Psi}{\partial \varphi}\,\mathrm{d}\varphi = 0,$$

where the first result depends on the expression of the Laplacian in polar coordinates, and the second is due to the derivative with respect to φ, which excludes the contribution of f_1.

As a consequence, an ordinary differential equation for f_2 can be derived from (3.2):

$$f_2'' + f_2'/r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{v}_0 \cdot \nabla\tau_0\,\mathrm{d}\varphi,$$

which implies

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rf_2'(r)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{v}_0 \cdot \nabla\tau_0\,\mathrm{d}\varphi.$$

Next, we can integrate over $r \in (r^*, 1+\mathscr{A}/2)$ and obtain

$$\begin{aligned} &2\pi(1+\mathscr{A}/2)f_2'(1+\mathscr{A}/2) \\ &\quad = \int_{r^*}^{1+\mathscr{A}/2} r\left(\int_{-\pi}^{\pi} \mathbf{v}_0 \cdot \nabla\tau_0\,\mathrm{d}\varphi\right)\mathrm{d}r. \end{aligned} \tag{3.3}$$

Furthermore, since Eqs. (1.3), (1.6) and (1.9) have the same structure, (3.3) holds true also by replacing τ_0 with τ and $\mathbf{v}_0$ with $\mathbf{v}$. Thus, if we write $\overline{Nu}_\delta(\tau) - \overline{Nu}_\delta(\tau_0)$, by adding and subtracting $\mathbf{v}_0 \cdot \nabla\tau$ inside the integral, it immediately follows that the general estimate for the error is

$$\begin{aligned} |\overline{Nu}_\delta(\tau) - \overline{Nu}_\delta(\tau_0)| &\leqslant \frac{k_p}{2\pi(1+\mathscr{A}/2)} \\ &\quad \times (\|\nabla\mathbf{u}\|\|\nabla\tau\| + \|\nabla\mathbf{v}_0\|\|\nabla\sigma\|). \end{aligned} \tag{3.4}$$

Now, we can substitute in (3.4) the estimates obtained in the previous section, so that the error is expressed in terms of Ra, $\mathscr{A}$ or b, and Pr.

Then, we immediately see (the proof is left to the reader) that if the approximation proposed for $\mathbf{v}$ and τ is working for large Pr the error in Nusselt number is small as well, while if the approximation is working for small Ra the error is even smaller.

Appendix A

Proof of (2.9). In this proof use will be made of Hölder's inequality concerning norms in L^p-spaces:

$$\int f_1 f_2 f_3 \leqslant \|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3} \quad \text{with}$$
$$p_1^{-1} + p_2^{-1} + p_3^{-1} = 1,$$

and of the embedding inequality for functions defined on 2-D domains (due to Ladyzenskaja, see [10] for a sketch of the proof):

$$\|f\|_4^2 \leqslant 2^{-1/2}\|f\|\|\nabla f\|,$$

where, for the sake of simplicity, it is understood that the norms without subscripts are in the Hilbert space.

In particular, we choose $f_1 = |\mathbf{v}|$, $f_2 = \tau$ and $f_3 = 1/r$, with $p_1 = p_2 = 4$ and $p_3 = 2$. Next, we apply Ladyzenskaja's inequality, and the chain of inequalities looks as follows:

$$\begin{aligned} \left|\int \frac{v_r}{r}\tau\right| &\leqslant \|\tau\|_4\|\mathbf{v}\|_4\left(2\pi\int_{\mathscr{A}/2}^{1+\mathscr{A}/2}\frac{1}{r^2}r\,\mathrm{d}r\right)^{1/2} \\ &\leqslant \sqrt{\pi b}\|\tau\|^{1/2}\|\mathbf{v}\|^{1/2}\|\nabla\tau\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}. \end{aligned}$$

Finally, (3.8) can immediately be deduced by applying also the Poincaré inequality. □

Proof of Proposition 2. The difference between the solution of (1.1)–(1.3) and that of (1.4)–(1.6) solves the homogeneous system:

$$\nabla \cdot \mathbf{u} = 0, \tag{A.1}$$

$$\begin{aligned} &-\Delta\mathbf{u} + \frac{1}{Pr}(\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v}_0) \\ &\quad + \frac{1}{Pr}(\mathbf{v}_0\cdot\nabla\mathbf{u} + \mathbf{v}_0\cdot\nabla\mathbf{v}_0) \\ &\quad = -\nabla(\Pi - \Pi_0) + Ra\,\sigma\nabla(r\sin\varphi), \end{aligned} \tag{A.2}$$

$$-\Delta\sigma + \mathbf{u}\cdot\nabla\tau_0 + \mathbf{u}\cdot\nabla\sigma + \mathbf{v}\cdot\nabla\sigma = \frac{u_r}{br}. \tag{A.3}$$

By repeating the procedure adopted in estimating the complete solution, we multiply (A.1) by $\mathbf{u}$, (A.2) by σ and integrate, to obtain

$$\begin{aligned} \int|\nabla\mathbf{u}|^2 &= \frac{1}{Pr}\Bigg(\underbrace{-\int\mathbf{u}\cdot\nabla\mathbf{v}_0\cdot\mathbf{u}}_{①}\underbrace{-\int\mathbf{v}_0\cdot\nabla\mathbf{v}_0\cdot\mathbf{u}}_{②}\Bigg) \\ &\quad + Ra\underbrace{\int\sigma\nabla(r\sin\varphi)\cdot\mathbf{u}}_{③}, \end{aligned} \tag{A.4}$$

$$\int|\nabla\sigma|^2 = \underbrace{-\int\mathbf{u}\cdot\nabla\tau_0\sigma}_{❶} + \underbrace{\int\frac{u_r}{rb}\sigma}_{❷}. \tag{A.5}$$

Next, use will again be made of Hölder's inequality and of the embedding inequality for functions defined on 2-D domains. Then, the estimates for the right-hand side of (A.4) are

① $$-\int \mathbf{u}\cdot\nabla\mathbf{v}_0\cdot\mathbf{u} \leqslant \left(\int |\nabla\mathbf{v}_0|^2\right)^{1/2}\left(\int |\mathbf{u}|^4\right)^{1/2} \overset{\text{Schwartz}}{\leqslant} \|\nabla\mathbf{v}_0\|\|\mathbf{u}\|_4^2 \overset{\text{Ladyzen}}{\leqslant} 2^{-1/2}\|\nabla\mathbf{v}_0\|\|\mathbf{u}\|\|\nabla\mathbf{u}\| \overset{\text{Poinc}}{\leqslant} 2^{-1/2}k_p\|\nabla\mathbf{v}_0\|\|\nabla\mathbf{u}\|^2,$$

② $$-\int \mathbf{v}_0\cdot\nabla\mathbf{v}_0\cdot\mathbf{u} \overset{\text{Hölder}}{\leqslant} \|\nabla\mathbf{v}_0\|_2\|\mathbf{v}_0\|_4\|\mathbf{u}\|_4 \overset{\text{Ladyzen}}{\leqslant} \|\nabla\mathbf{v}_0\|_2 2^{-1/2}\|\nabla\mathbf{v}_0\|_2^{1/2}\|\mathbf{v}_0\|_2^{1/2}\|\nabla\mathbf{u}\|_2^{1/2}\|\mathbf{u}\|_2^{1/2} \overset{\text{Ponic}}{\leqslant} 2^{-1/2}\|\nabla\mathbf{v}_0\|\|\nabla\mathbf{v}_0\|^{1/2}k_p^{1/2} \times \|\nabla\mathbf{v}_0\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}k_p^{1/2}\|\nabla\mathbf{u}\|^{1/2} = 2^{-1/2}k_p\|\nabla\mathbf{v}_0\|^2\|\nabla\mathbf{u}\|,$$

③ $$\int \sigma\mathbf{e}_z\cdot\mathbf{u} \leqslant |\sigma|\|\mathbf{u}\| \leqslant k_p^2\|\nabla\sigma\|\|\nabla\mathbf{u}\|.$$

Those for (A.5) are

❶ $$-\int \mathbf{u}\cdot\nabla\tau_0\sigma \overset{②}{\leqslant} 2^{-1/2}k_p\|\nabla\sigma\|\|\nabla\mathbf{u}\|\|\nabla\tau_0\|,$$

❷ $$\int \frac{u_r}{rb}\sigma \leqslant \frac{2k_p^2}{\mathscr{A}b}\|\nabla\mathbf{u}\|\|\nabla\sigma\|.$$

All this can be summarized as follows:

$$\|\nabla\mathbf{u}\| \leqslant \frac{k_p}{2^{1/2}Pr}(\|\nabla\mathbf{v}_0\|\|\nabla\mathbf{u}\| + \|\nabla\mathbf{v}_0\|^2) + k_p^2 Ra\|\nabla\sigma\|, \tag{A.6}$$

$$\|\nabla\sigma\| \leqslant \frac{k_p}{2^{1/2}}\|\nabla\tau_0\|\|\nabla\mathbf{u}\| + \frac{2k_p^2}{\mathscr{A}b}\|\nabla\mathbf{u}\|. \tag{A.7}$$

Here, if one had used (2.9) the second estimate could be substituted by

$$\|\nabla\sigma\| \leqslant \frac{k_p}{2^{1/2}}\|\nabla\tau_0\|\|\nabla\mathbf{u}\| + \sqrt{\frac{\pi}{b}}k_p\|\nabla\mathbf{u}\|. \tag{A.8}$$

Next, the estimates of Proposition 1 can be applied to the solution $(\mathbf{v}_0, \tau_0)$ of the non-linear Stokes problem, and such estimates substituted in (A.6) and (A.7) or (A.8). Finally, by rearranging the terms in an elementary manner, the statement of Proposition 2 follows. □

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